Compute high-pass and low-pass filters

Review: high and low frequency components of DFT.

1D case: given $f(x)$, $x = 0, 1, \cdots, N-1$.

$$\hat{f}(t) = \frac{1}{N} \sum_{k=0}^{N-1} f(x)\, e^{-2\pi j \frac{kt}{N}}$$

$$f(x) = \sum_{l=0}^{N-1} \hat{f}(l)\, e^{2\pi j \frac{lx}{N}}$$

$$\vec{f} = \sum_{l=0}^{N-1} \hat{f}(l)\, \vec{u}_l$$

The DFT eigenvectors $\vec{u}_l$ are:

$$\vec{u}_l = \left(e^{2\pi j \frac{lx}{N}} \right)_{x=0}^{N-1} = \begin{pmatrix} 1 \\ e^{2\pi j \frac{l}{N}} \\ e^{2\pi j \frac{2l}{N}} \\ \vdots \\ e^{2\pi j \frac{(N-1)l}{N}} \end{pmatrix} \quad \text{if } l \le \frac{N}{2}.$$

$$\vec{u}_l = \left(e^{2\pi j \frac{lx}{N}} \right)_{x=0}^{N-1} = \left(e^{2\pi j \left(\frac{lx}{N} - \frac{Nx}{N} \right)} \right)_{x=0}^{N-1} = \left(e^{2\pi j \frac{(l-N)x}{N}} \right)_{x=0}^{N-1}$$

$$= \begin{pmatrix} 1 \\ e^{2\pi j \frac{l-N}{N}} \\ e^{2\pi j \frac{2(l-N)}{N}} \\ \vdots \\ e^{2\pi j \frac{(N-1)(l-N)}{N}} \end{pmatrix} \quad \text{if } l > \frac{N}{2}.$$

So, $\vec{u}_l$ corresponds to a low frequency component if l is close to 0 or N.

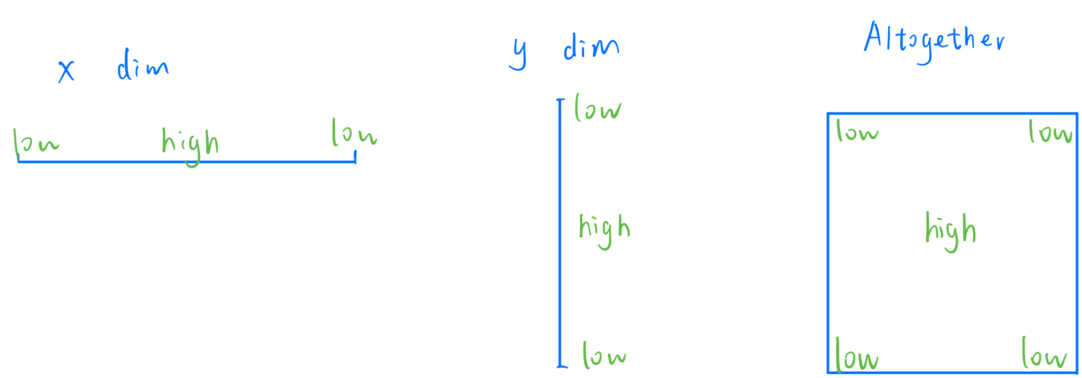
high frequency component if l is close to $\frac{N}{2}$.

The 2D case is similar but we need to consider frequency in 2 dimensions

$$
\begin{aligned}
F(m,n) = & \sum_{0\le k,l\le \frac{N}{2}-1} \left[\hat{F}\left(\frac{N}{2}+k, \frac{N}{2}+l\right) e^{j\frac{2\pi}{N}[(\frac{N}{2}+k)m+(\frac{N}{2}+l)n]}\right] \\
& + \sum_{1\le k,l\le \frac{N}{2}-1} \left[\overline{\hat{F}\left(\frac{N}{2}+k, \frac{N}{2}+l\right)} e^{j\frac{2\pi}{N}[(\frac{N}{2}-k)m+(\frac{N}{2}-l)n]}\right] \\
& + \sum_{0\le k,l\le \frac{N}{2}-1} \left[\hat{F}\left(\frac{N}{2}+k, \frac{N}{2}-l\right) e^{j\frac{2\pi}{N}[(\frac{N}{2}+k)m+(\frac{N}{2}-l)n]}\right] \\
& + \sum_{1\le k,l\le \frac{N}{2}-1} \left[\overline{\hat{F}\left(\frac{N}{2}+k, \frac{N}{2}-l\right)} e^{j\frac{2\pi}{N}[(\frac{N}{2}-k)m+(\frac{N}{2}+l)n]}\right] \\
& + \sum_{0\le l\le \frac{N}{2}-1} \hat{F}\left(0, \frac{N}{2}+l\right) e^{j\frac{2\pi}{N}[(\frac{N}{2}+l)n]} + \sum_{1\le l\le \frac{N}{2}-1} \overline{\hat{F}\left(0, \frac{N}{2}+l\right)} e^{j\frac{2\pi}{N}[(\frac{N}{2}-l)n]} \\
& + \sum_{0\le k\le \frac{N}{2}-1} \hat{F}\left(\frac{N}{2}+k, 0\right) e^{j\frac{2\pi}{N}[(\frac{N}{2}+k)m]} + \sum_{1\le k\le \frac{N}{2}-1} \overline{\hat{F}\left(\frac{N}{2}+k, 0\right)} e^{j\frac{2\pi}{N}[(\frac{N}{2}-k)m]} + \hat{F}(0,0)
\end{aligned}
$$

DFT is separable

2D DFT = 2 x 1D DFT



Review :

high-pass filter: output high-frequency components

low-pass filter: output low-frequency components

Compute high and low-pass filters without assuming central spectrum.

1. Given an $M \times N$ image f with indices $0 \le m \le M-1$, $0 \le n \le N-1$, first compute its DFT F with indices $0 \le m \le M-1$, $0 \le n \le N-1$,

2. Construct a $M \times N$ filter matrix H.

   set the indices of H to be

$$\begin{cases} -\frac{M}{2} \le m \le \frac{M}{2} - 1, \; -\frac{N}{2} \le n \le \frac{N}{2} - 1 & \text{if } M, N \text{ even} \\ -\frac{M}{2} \le m \le \frac{M}{2} - 1, \; -\frac{N-1}{2} \le n \le \frac{N-1}{2} & \text{if } M \text{ even}, N \text{ odd} \\ -\frac{M-1}{2} \le m \le \frac{M-1}{2}, \; -\frac{N}{2} \le n \le \frac{N}{2} - 1 & \text{if } M \text{ odd}, N \text{ even} \\ -\frac{M-1}{2} \le m \le \frac{M-1}{2}, \; -\frac{N-1}{2} \le n \le \frac{N-1}{2} & \text{if } M, N \text{ odd} \end{cases}$$

   Define $H(m,n) = h\left(|(m,n)|\right)$, $|(m,n)| = \sqrt{m^2+n^2}$

   where $h(x)$ is prescribed according to which task we are performing.

3. Translate the matrix H so that the indices of H are

   $0 \le m \le M-1$, $0 \le n \le N-1$.

Ideal high pass filter, $h(x) = \begin{cases} 1 & \text{if } 0 \le x \le D_0^2 \\ 0 & \text{if } x > D_0^2 \end{cases}$

Ideal low pass filter, $h(x) = \begin{cases} 0 & \text{if } 0 \le x \le D_0^2 \\ 1 & \text{if } x > D_0^2 \end{cases}$

Gaussian high-pass filter: $h(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right)$

Gaussian low-pass filter: $h(x) = 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right)$

Examples:

Compute ideal low / high-pass filters of radius 2 for

Gaussian high-pass filter with $\sigma = 1$

(a) 4x4 images (b) 5x5 images

(a)

$$\begin{array}{c|cccc} & -2 & -1 & 0 & 1 \\ \hline -2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{array}$$

ideal high-pass filter.

$H(m,n) = \begin{cases} 1 & \text{if } m^2 + n^2 \le 2 \\ 0 & \text{otherwise} \end{cases}$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 3 & 0 & 1 & 1 & 1 \end{array}$$

$$
\begin{array}{c|cccc}
 & -2 & -1 & 0 & 1 \\
\hline
-2 & 1 & 1 & 1 & 1 \\
-1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0
\end{array}
$$

ideal low-pass filter

$$
H(m,n) = \begin{cases} 0 & \text{if } m^2+n^2 \le 2 \\ 1 & \text{otherwise} \end{cases}
$$

$$\downarrow$$

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 \\
3 & 1 & 0 & 0 & 0
\end{array}
$$

$$
\begin{array}{c|cccc}
 & -2 & -1 & 0 & 1 \\
\hline
-2 & e^{-4} & e^{-\frac{5}{2}} & e^{-2} & e^{-\frac{5}{2}} \\
-1 & e^{-\frac{5}{2}} & e^{-1} & e^{-\frac{1}{2}} & e^{-1} \\
0 & e^{-2} & e^{-\frac{1}{2}} & 1 & e^{-\frac{1}{2}} \\
1 & e^{-\frac{5}{2}} & e^{-1} & e^{-\frac{1}{2}} & e^{-1}
\end{array}
$$

Gaussian high-pass filter

$$
H(m,n) = \exp\left(-\frac{m^2+n^2}{2}\right)
$$

$$\downarrow$$

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & e^{-4} & e^{-\frac{5}{2}} & e^{-2} & e^{-\frac{5}{2}} \\
1 & e^{-\frac{5}{2}} & e^{-1} & e^{-\frac{1}{2}} & e^{-1} \\
2 & e^{-2} & e^{-\frac{1}{2}} & 1 & e^{-\frac{1}{2}} \\
3 & e^{-\frac{5}{2}} & e^{-1} & e^{-\frac{1}{2}} & e^{-1}
\end{array}
$$

(b)

	-2	-1	0	1	2
-2	0	0	0	0	0
-1	0	1	1	1	0
0	0	1	1	1	0
1	0	1	1	1	0
2	0	0	0	0	0

ideal high-pass filter,

$$H(m,n) = \begin{cases} 1 & \text{if } m^2+n^2 \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$\Downarrow$

	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	0
2	0	1	1	1	0
3	0	1	1	1	0
4	0	0	0	0	0

ideal low-pass filter

	0	1	2	3	4
0	1	1	1	1	1
1	1	0	0	0	1
2	1	0	0	0	1
3	1	0	0	0	1
4	1	1	1	1	1

	-2	-1	0	1	2
-2	e^{-4}	$e^{-\frac{5}{2}}$	e^{-2}	$e^{-\frac{5}{2}}$	e^{-4}
-1	$e^{-\frac{5}{2}}$	e^{-1}	$e^{-\frac{1}{2}}$	e^{-1}	$e^{-\frac{5}{2}}$
0	e^{-2}	$e^{-\frac{1}{2}}$	1	$e^{-\frac{1}{2}}$	e^{-2}
1	$e^{-\frac{5}{2}}$	e^{-1}	$e^{-\frac{1}{2}}$	e^{-1}	$e^{-\frac{5}{2}}$
2	e^{-4}	$e^{-\frac{5}{2}}$	e^{-2}	$e^{-\frac{5}{2}}$	e^{-4}

Gaussian high-pass filter,

$$H(m,n) = \exp\left(-\frac{m^2+n^2}{2}\right)$$

$$\Downarrow$$

	0	1	2	3	4
0	e^{-4}	$e^{-\frac{5}{2}}$	e^{-2}	$e^{-\frac{5}{2}}$	e^{-4}
1	$e^{-\frac{5}{2}}$	e^{-1}	$e^{-\frac{1}{2}}$	e^{-1}	$e^{-\frac{5}{2}}$
2	e^{-2}	$e^{-\frac{1}{2}}$	1	$e^{-\frac{1}{2}}$	e^{-2}
3	$e^{-\frac{5}{2}}$	e^{-1}	$e^{-\frac{1}{2}}$	e^{-1}	$e^{-\frac{5}{2}}$
4	e^{-4}	$e^{-\frac{5}{2}}$	e^{-2}	$e^{-\frac{5}{2}}$	e^{-4}